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On the Stability and Controller Robustness of Some Popular PID Tuning Rules

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***Abstract*—In this note, we study the stability and controller robustness of some popular proportional-integral-derivative (PID) tuning techniques that are based on first-order models with time delays. Using the characterization of all stabilizing PID controllers derived in a previous paper, each tuning rule is analyzed to first determine if the proportional gain value dictated by that rule, lies inside the range of admissible proportional gains. Then, the integral and derivative gain values are examined to determine conditions under which the tuning rule exhibits robustness with respect to controller parameter perturbations.**

***Index Terms*—Controller robustness, proportional-integral-derivative (PID) controllers, stability, tuning rules.**

I. Introduction

Over the last forty years, numerous methods have been developed for setting the parameters of a proportional-integral-derivative (PID) controller [1]. Some of these methods are based on characterizing the dynamic response of the plant to be controlled with a first-order model with time delay. Traditionally, this model is obtained by applying a step input to the plant and measuring at the output the following three parameters: the steady-state gain, the time constant, and the time delay. Although many of these tuning techniques work in practice, not much is known about the robustness or stability of these algorithms beyond what has been observed in empirical studies. Perhaps, the only exception is the internal model control (IMC) algorithm where the stability constraint is built into the PID design method. Recent results on PID stabilization obtained in [6], however, make it possible to revisit these classical tuning rules and to justify them in terms of stability and robustness. The main objective of this note is to do precisely that.

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In this note, we will analyze several PID tuning techniques that are based on first-order models with time delay. This analysis will attempt to describe when each tuning technique is appropriate in the sense of providing PID controller parameters that are robust in the space of the controller coefficients. A controller for which the closed-loop system is destabilized by small perturbations in the controller coefficients is said to be *fragile* [4]. Any controller that is to be practically implemented must necessarily be nonfragile or *controller robust* (terminology suggested by W. M. Wonham) [7] so that: 1) round-off errors during implementation do not destabilize the closed-loop; and 2) tuning of the parameters about the nominal design values is allowed.

Four tuning techniques will be discussed: 1) the classical Ziegler–Nichols step response method; 2) the CHR method; (3) the Cohen–Coon method; and 4) the IMC design technique. The analysis starts by ascertaining if the proposed proportional gain value lies inside the allowable range determined in [6]. We will then examine for this fixed proportional gain, the location of the integral and derivative gain values inside the stability region described in [6]. This procedure will allow us to determine conditions under which each tuning technique provides a good l_2 parametric stability margin in the space of the controller coefficients. In this way, we will avoid undesirable scenarios such as PID controller parameters that are dangerously close to instability.

The note is organized as follows. In Section II, we recall some recent results on PID stabilization of first-order plants with time-delay [6]. These results are used in Section III to analyze the Ziegler–Nichols step response method. Section IV summarizes the results of similar analyzes for the other three methods. Finally, Section V contains some concluding remarks.

II. Preliminary Results

The tuning techniques analyzed in this note are based on characterizing the plant to be controlled by the following transfer function

$$G(s) = \frac{k}{1+Ts}e^{-Ls} \tag{1}$$

where k is the steady-state gain, L is the apparent time delay, and T is the apparent time constant. We will consider the feedback control system shown in Fig. 1 where r is the command signal, y is the output of the plant, $G(s)$ given by (1) is the plant to be controlled, and $C(s)$ is the controller. We focus on the case when the controller is of the PID type, i.e., the controller has a proportional term, an integral term and a derivative term. There are different ways of representing the PID control algorithm [1]. In our case, we will use the following representation:

$$C(s) = k_p + \frac{k_i}{s} + k_d s$$

where k_p is the proportional gain, k_i is the integral gain and k_d is the derivative gain.

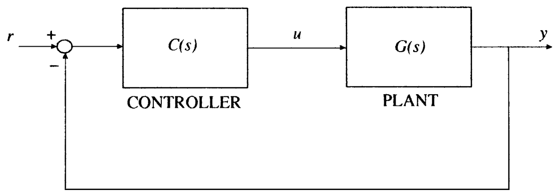

Fig. 1. Feedback control system.

The following theorem [6] provides an analytical characterization of the set of controller parameters (k_p, k_i, k_d) for which the closed-loop system in Fig. 1 is stable.

Theorem 2.1: The range of k_p values for which a given open-loop stable plant, with transfer function $G(s)$ as in (1), continues to have closed-loop stability with a PID controller in the loop is given by

$$-\frac{1}{k} < k_p < \frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right] \tag{2}$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\alpha \tag{3}$$

in the interval $(0, \pi)$. For k_p values outside this range, there are no stabilizing PID controllers. The complete stabilizing region is given by (see Fig. 2) the following.

1) For each $k_p \in (-(1/k), (1/k))$, the cross section of the stabilizing region in the (k_i, k_d) space is the trapezoid T.
2) For $k_p = (1/k)$, the cross-section of the stabilizing region in the (k_i, k_d) space is the triangle Δ.
3) For each $k_p \in ((1/k), k_{\text{upp}} := (1/k)[(T/L)\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)])$, the cross-section of the stabilizing region in the (k_i, k_d) space is the quadrilateral Q.

In Fig. 2, the parameters m_j, b_j, and w_j, for $j = 1, 2$, are defined as follows:

$$\begin{aligned} m_j &\triangleq \frac{L^2}{z_j^2} \\ b_j &\triangleq -\frac{L}{kz_j}\left[\sin(z_j) + \frac{T}{L}z_j\cos(z_j)\right] \\ w_j &\triangleq \frac{z_j}{kL}\left[\sin(z_j) + \frac{T}{L}z_j(\cos(z_j)+1)\right] \end{aligned} \tag{4}$$

where $z_1, z_2, z_2 > z_1$ are the solutions of

$$kk_p + \cos(z) - \frac{T}{L}z\sin(z) = 0$$

in the interval $(0, \pi)$.

III. Ziegler–Nichols Step Response Method

A simple way to determine the parameters of a PID controller based on step response data was developed by Ziegler and Nichols in 1942 [8]. This method first characterizes the plant by the parameters L and a, where the parameter a is defined as

$$a = k\frac{L}{T}.$$

Once these parameters are determined, the PID controller parameters are then given in terms of L and a by the following formulas:

$$\overline{k_p} = \frac{1.2}{a} \quad \overline{k_i} = \frac{0.6}{aL} \quad \overline{k_d} = \frac{0.6L}{a}. \tag{5}$$

This tuning rule was developed by empirical simulations of many different systems and is only applicable to open-loop *stable* plants. We

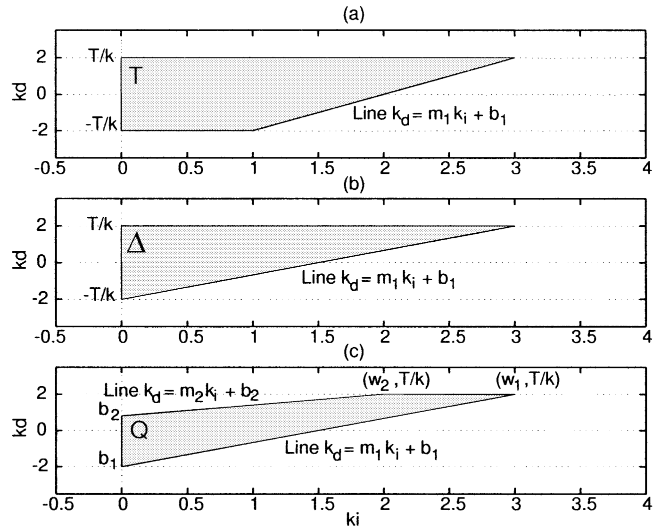

Fig. 2. Stabilizing region of (k_i, k_d) for: (a) $-(1/k) < k_p < (1/k)$; (b) $k_p = (1/k)$; and (c) $(1/k) < k_p < k_{\text{upp}}$.

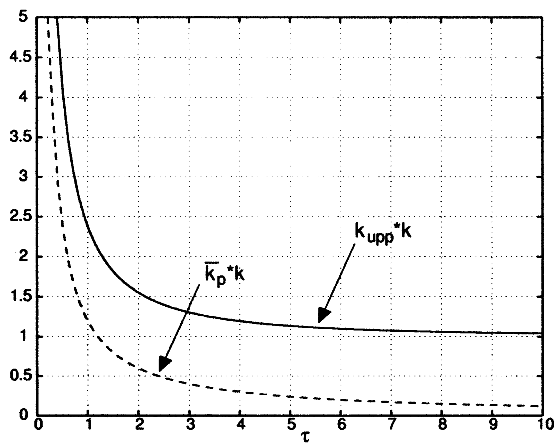

Fig. 3. Comparison of $\overline{k_p}$ given by the Ziegler-Nichols Method and the upper bound k_{upp}.

now define the parameter τ as the ratio of the apparent time delay to the apparent time constant of the plant, i.e.,

$$\tau = \frac{L}{T}.$$

First, we consider the proportional gain value given in (5) and rewrite it as a function of τ

$$\overline{k_p} = \frac{1.2}{k\tau}. \tag{6}$$

Since $k > 0$ and $\tau > 0$ (the plant is open-loop stable), then $\overline{k_p} > 0$. From Theorem 2.1, we can rewrite the upper bound on k_p as a function of the parameter τ

$$k_{\text{upp}} = \frac{1}{k}\left[\frac{1}{\tau}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right] \tag{7}$$

where α_1 is now the solution of the equation

$$\tan(\alpha) = -\frac{1}{1+\tau}\alpha$$

in the interval $(0, \pi)$. We now compare $\overline{k_p}$ and k_{upp} by plotting $\overline{k_p}k$ and $k_{\text{upp}}k$ as functions of the parameter τ. As can be seen from Fig. 3, the proportional gain value given by the Ziegler-Nichols step response method is always less than the upper bound k_{upp}. Thus, this tuning technique always provides a feasible proportional gain value $\overline{k_p}$. We

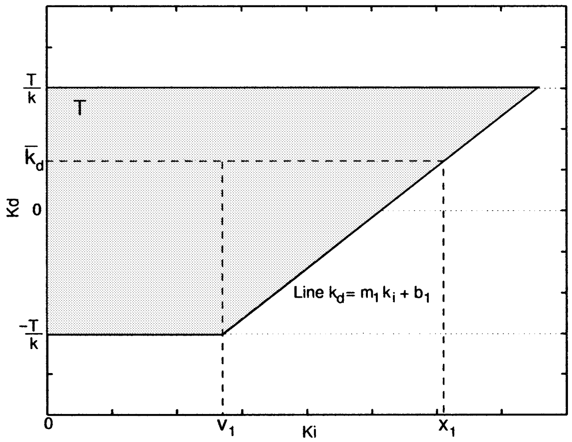

Fig. 4. Location of the parameters $(\overline{k_i}, \overline{k_d})$ when $\tau \geq 1.2$.

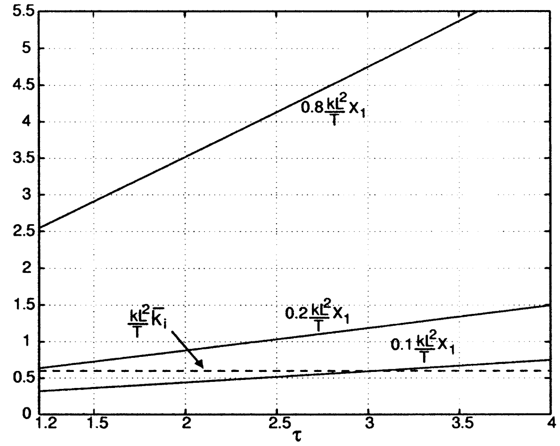

Fig. 5. Comparison of $0.2(kL^2/T)x_1$, $0.8(kL^2/T)x_1$, and $(kL^2/T)\overline{k_i}$ for $\tau \geq 1.2$.

now set $k_p = \overline{k_p}$ and consider two cases, requiring different treatment according to the results presented in Section II. Moreover, for clarity of presentation, let us rewrite the parameters $\overline{k_i}$ and $\overline{k_d}$ in (5) as

$$\overline{k_i} = \frac{0.6T}{kL^2} \tag{8}$$

$$\overline{k_d} = \frac{0.6T}{k}. \tag{9}$$

Case 1: $\tau \geq 1.2$. In this case, we have $0 < \overline{k_p} \leq (1/k)$. Then the stabilizing set is given either by Fig. 2(a) or by Fig. 2(b). Notice from (9) that the parameter $\overline{k_d}$ is always less than (T/k) as illustrated in Fig. 4. The derivative gain value provided by the Ziegler–Nichols method is robust in the sense that it is not close to the stability boundary (T/k). Following the same principle, we would like to guarantee that the integral gain value is also far away from the stability boundary. Let x_1 be the k_i-coordinate of the point where the line $k_d = \overline{k_d}$ intersects the line $k_d = m_1 k_i + b_1$. From Fig. 4, we now find the conditions under which the parameter $\overline{k_i}$ lies in the range $(0.2x_1, 0.8x_1)$. Following the same derivation used in [6], x_1 can be expressed as follows

$$x_1 = \frac{T}{kL^2} z_1[\tau \sin(z_1) + z_1(\cos(z_1) + 0.6)] \tag{10}$$

where z_1 is the solution of

$$k\overline{k_p} + \cos(z) - \frac{T}{L} z \sin(z) = 0$$
$$\Leftrightarrow 1.2 + \tau \cos(z) - z \sin(z) = 0 \quad [\text{using } (6)]$$

in the interval $(0, \pi)$. From (8) and (10), we can plot the terms $(kL^2/T)x_1$, and $(kL^2/T)\overline{k_i}$ versus τ. This graph is shown in Fig. 5 for $\tau \geq 1.2$. As can be seen from this graph, $\overline{k_i}$ does not lie in the range $(0.2x_1, 0.8x_1)$ for any value of τ. If we relax our robustness condition and now make $\overline{k_i}$ lie inside the range $(0.1x_1, 0.8x_1)$, we see from Fig. 5 that this occurs for $1.2 \leq \tau < 3$. In this way, for $1.2 \leq \tau < 3, \overline{k_i}$ will be located 10% of x_1 away from the k_d-axis which corresponds to a good l_2 parametric stability margin.

Case 2: $0 < \tau < 1.2$. In this case, we have $(1/k) < \overline{k_p} < k_{\text{upp}}$. The stabilizing set is given by Fig. 2(c). We now show that the parameter $\overline{k_d}$ is less than b_2 for all $\tau < 1.2$. From (4), b_2 can be rewritten as follows:

$$b_2 = -\frac{T}{k}\left[\tau \frac{\sin(z_2)}{z_2} + \cos(z_2)\right]$$

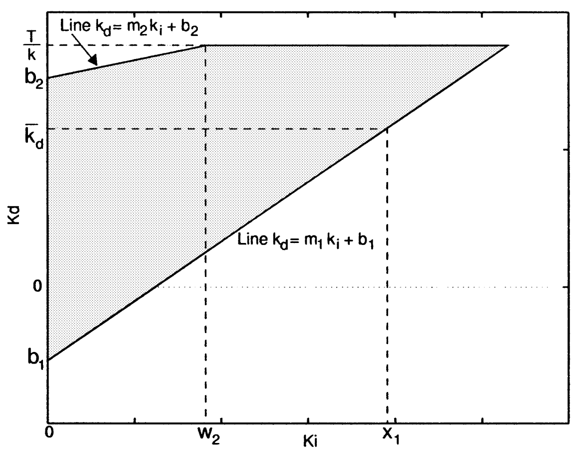

Fig. 6. Location of the parameters $(\overline{k_i}, \overline{k_d})$ when $0 < \tau < 1.2$.

where $z_2 > z_1 > 0$ is the solution of

$$1.2 + \tau \cos(z) - z \sin(z) = 0$$

in the interval $(0, \pi)$. By sweeping τ in the range $(0, 1.2)$, it can be shown that $\overline{k_d} < b_2$. Fig. 6 shows the location of $\overline{k_d}$ with respect to the stabilizing set in the space of (k_i, k_d).

As in the previous case, we will now analyze for which values of τ, the parameter $\overline{k_i}$ lies inside the range $(0.2x_1, 0.8x_1)$. As in Case 1, we can plot the terms $0.2(kL^2/T)x_1$, $0.8(kL^2/T)x_1$, and $(kL^2/T)\overline{k_i}$ versus τ. This graph is shown in Fig. 7 for $0 < \tau < 1.2$. From this graph we see that $\overline{k_i}$ lies in the range $(0.2x_1, 0.8x_1)$ for $0 < \tau < 1.07$. For the relaxed condition where $\overline{k_i}$ lies in the range $(0.1x_1, 0.8x_1)$, we have $0 < \tau < 1.2$.

From the previous analysis, we conclude that the Ziegler–Nichols step response method gives a controller-robust PID controller for $0 < \tau < 1.07$. Controller robustness is here understood as good parametric stability margin in the space of (k_i, k_d).

Remark 3.1: It has been determined empirically [1] that the Ziegler–Nichols rule is applicable if $0.1 < \tau < 0.6$. In this range, the derivative action often gives significant improvement of performance. Comparing this range with the one previously obtained for controller robustness, we see that the former is included in the latter. Thus, for $0.1 < \tau < 0.6$, the Ziegler–Nichols step response method not only gives good performance but also is robust with respect to controller parameter perturbations.

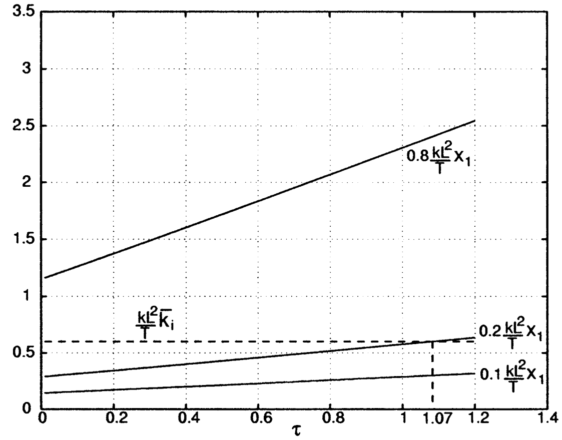

Fig. 7. Comparison of $0.2(kL^2/T)x_1$, $0.8(kL^2/T)x_1$, and $(kL^2/T)\overline{k_i}$ for $0 < \tau < 1.2$.

IV. OTHER TUNING TECHNIQUES

The analysis presented in the previous section can be applied to other PID tuning techniques that are based on first-order models with time delays. The main criterion is to ensure first that the controller parameters k_p and k_d are inside the stabilizing set of gain values. Then, the parameter k_i is forced to lie inside an interval located 20% of x_1 away from the boundaries of the stabilizing set in the (k_i, k_d) space. Here, x_1 represents the maximum stabilizing integral gain value for the fixed proportional and derivative gains provided by the particular tuning rule. As a result of this criterion, the range of (L/T) values that ensures controller robustness can be determined for each tuning technique. These values are summarized as follows:

$$\begin{aligned} \text{Ziegler–Nichols Step Response Method}: 0 < \frac{L}{T} < 1.07 \\ \text{Cohen–Coon Method [3]}: 0 < \frac{L}{T} < 8.53 \\ \text{CHR Method [2]}: 0.37 < \frac{L}{T} \\ \text{IMC Design Technique (for } \lambda/L = 0.25\text{) [5]}: 0.37 < \frac{L}{T}. \end{aligned}$$

From this table, we conclude that the Cohen–Coon method gives resilient PID parameters in the sense of the parametric stability margin when the plant under study satisfies the property $0 < (L/T) < 8.53$.

It is interesting to note that for both the CHR method and the IMC Design Technique the same resilience of the PID parameters is obtained if the ratio (L/T) is greater than a lower bound, which is 0.37. In the case of the IMC Design Technique, the design variable $\lambda > 0$ should be selected properly in order to obtain a PID controller with a good compromise between performance and robustness. It is commonly recommended [8] that λ/L should be fixed at 0.25, which was the value used in the above table.

V. CONCLUDING REMARKS

In this note, we have presented an analysis of the robustness of some common PID tuning techniques in the space of the controller parameters. This analysis was motivated by the fact that a good PID controller design should exhibit robustness with respect to small perturbations in the controller coefficients. Since the results of [6] yield a *complete* characterization of all stabilizing PID controllers for a particular class of plants, it is clear that in principle a similar robustness analysis with respect to plant parameter perturbations is also possible. The details, however, remain to be worked out.

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Discretization Behaviors of Equivalent Control Based Sliding-Mode Control Systems

Xinghuo Yu and Guanrong Chen

***Abstract*—In this note, discretization behaviors of the equivalent control based sliding-mode control (SMC) systems are studied. Some inherent dynamical properties of the discretized second-order systems are first explored. Upper bounds for the system steady states are established. The system's steady-state behaviors are discussed. The analysis for the second-order systems is then extended to higher order systems. Simulations are presented to verify the theoretical results.**

***Index Terms*—Discretization, dynamical behavior, equivalent control, sliding-mode control.**

I. INTRODUCTION

Discrete sliding-mode control (DSMC) has been extensively studied to address some basic problems associated with the SMC of discrete-time systems that have relatively low switching frequencies. Major research efforts in DSMC have been devoted to the development of various controllers using specific guiding principles [1]–[9]. However, the study of discretizing a continuous-time SMC for digital implementation has not been fully explored.

In this note, we study the discretization behaviors of the most popular SMC systems—the equivalent control based SMC systems. Unlike most existing research on DSMC, our present interest is in the discretization effect on a continuous-time SMC system.

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