New Results on the Synthesis of PID Controllers

Guillermo J. Silva, Aniruddha Datta, and S. P. Bhattacharyya

***Abstract*—This paper considers the problem of stabilizing a first-order plant with dead-time using a proportional-integral-derivative (PID) controller. Using a version of the Hermite–Biehler Theorem applicable to quasipolynomials, the complete set of stabilizing PID parameters is determined for both open-loop stable and unstable plants. The range of admissible proportional gains is first determined in closed form. For each proportional gain in this range the stabilizing set in the space of the integral and derivative gains is shown to be either a trapezoid, a triangle or a quadrilateral. For the case of an open-loop unstable plant, a necessary and sufficient condition on the time delay is determined for the existence of stabilizing PID controllers.**

***Index Terms*—Closed-loop systems, delay systems, proportional-integral-derivative (PID) control, reduced-order systems, stability.**

I. Introduction

The majority of control systems in the world are operated by proportional-integral-derivative (PID) controllers. Indeed, it has been reported that 98% of the control loops in the pulp and paper industries are controlled by single-input single-output PI controllers [1] and that in process control applications, more than 95% of the controllers are of the PID type [2]. Similar statistics hold in the motion control and aerospace industries.

Given the widespread industrial use of PID controllers, it is clear that even a small percentage improvement in PID design could have a tremendous impact worldwide. Despite this, it is unfortunate that currently there is not much theory dealing with PID designs. Indeed, most of the industrial PID designs are still carried out using only empirical techniques and the mathematically elegant and sophisticated theories developed in the context of modern optimal control cannot be applied to them. This represents a significant gap between the theory and practice of automatic control.

Over the last four decades, numerous methods have been developed for setting the parameters of P, PI, and PID controllers. Some of these methods are based on characterizing the dynamic response of the plant to be controlled with a first-order model with time delay. It is interesting to note that even though most of these tuning techniques by and large provide satisfactory results, the set of all stabilizing PID controllers for these first-order models with dead time remains unknown. This fact motivated the present paper. Our objective was to provide a complete solution to the problem of characterizing the set of all PID gains that stabilize a given first-order plant with time delay.

In earlier work [3], a generalization of the Hermite–Biehler Theorem was derived and then used to compute the set of all stabilizing PID controllers for a given linear, time invariant plant described by a *rational* transfer function. The approach developed in [3] constitutes the first attempt to find a characterization of all stabilizing PID controllers for a given plant. However, the synthesis results presented in that reference cannot be applied directly to plants containing time delays since they were obtained for plants described by *rational transfer functions*.

Plants with time delays give rise to characteristic equations containing quasipolynomials. Our approach in this paper will be to make use of a version of the Hermite–Biehler Theorem applicable to quasipolynomials. Such a result was derived by Pontryagin in [4]. Pontryagin's theorems have been earlier used to develop graphical criteria to study the stability of systems with time delays (see [5] and the references therein). These references dating back to the 1960s represent the substantial progress made by several researchers in studying the stability of time delay systems with up to *two variable parameters*. The PID problem considered in this paper, however, involves three adjustable parameters and is considerably more difficult.

The solution to the PID stabilization problem presented here is based on first determining the range of the proportional parameter for which a stabilizing PID controller exists. Then, for a fixed value of the proportional parameter in that range, it is shown that the set of stabilizing integral and derivative gain values lie in a convex polygon (either a trapezoid, a triangle or a quadrilateral). Furthermore, the boundaries of these polygons can be generated using straightforward computations. The results of this paper show that, contrary to popular belief, there is a need for substantial theory in PID control. Moreover, it is our belief that this is precisely the direction of research needed to close the theory-practice gap that has arisen in the control field.

II. Main Results

Systems with step responses like the one shown in Fig. 1 are commonly modeled as first order processes with a time delay [2], and can be mathematically described by

$$G(s) = \frac{k}{1+Ts}\, e^{-Ls} \tag{2.1}$$

where k represents the steady-state gain of the plant, L represents the time delay, and T represents the time constant of the plant.

Consider now the feedback control system shown in Fig. 2 where r is the command signal, y is the output of the plant, $G(s)$ given by (2.1) is the plant to be controlled, and $C(s)$ is the

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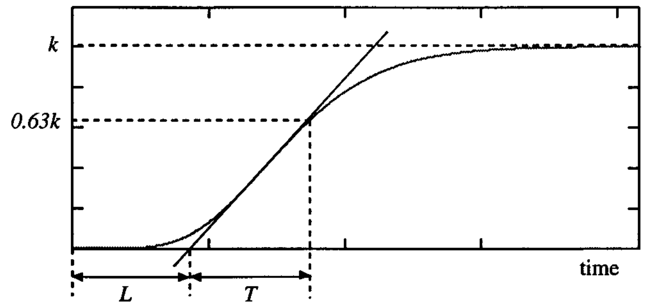

Fig. 1. Open-loop step response.

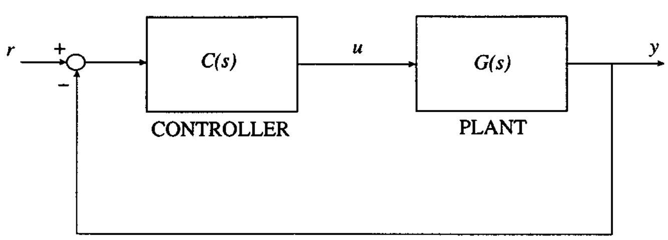

Fig. 2. Feedback control system.

controller. In this paper, we focus on the case when the controller is of the PID type, i.e.,

$$C(s) = k_p + \frac{k_i}{s} + k_d s.$$

The objective is to determine the set of controller parameters (k_p, k_i, k_d) for which the closed-loop system is stable.

Next, we state the main results of this paper.

A. Open-Loop Stable Plant

In this case $T > 0$. Furthermore, we make the standing assumption that $k > 0$ and $L > 0$.

Theorem 2.1: The range of k_p values for which a given open-loop stable plant, with transfer function $G(s)$ as in (2.1), can be stabilized using a PID controller is given by

$$-\frac{1}{k} < k_p < \frac{1}{k}\left[\frac{T}{L}\,\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right] \tag{2.2}$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\,\alpha \tag{2.3}$$

in the interval $(0, \pi)$. For k_p values outside this range, there are no stabilizing PID controllers. The complete stabilizing region is given by: (see Fig. 3)

1) For each $k_p \in (-(1/k), 1/k)$, the cross-section of the stabilizing region in the (k_i, k_d) space is the trapezoid T.
2) For $k_p = 1/k$, the cross-section of the stabilizing region in the (k_i, k_d) space is the triangle Δ.
3) For each $k_p \in (1/k, k_u := 1/k[(T/L)\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)])$, the cross-section of the stabilizing region in the (k_i, k_d) space is the quadrilateral Q.

The parameters m_j, b_j, w_j, $j = 1, 2$ necessary for determining the boundaries of T, Δ and Q can be determined using equations (4.13), (4.14), (4.10), (4.11), (4.18) where z_j, $j = 1, 2, \ldots$ are the positive-real solutions of (4.6) arranged in ascending order of magnitude.

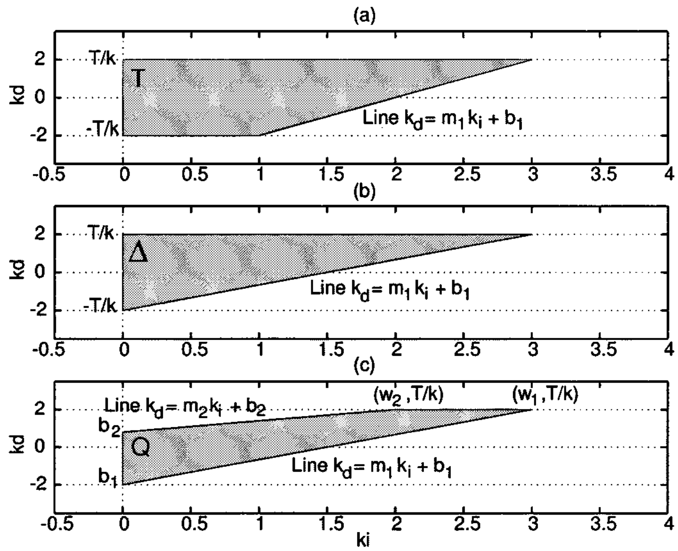

Fig. 3. The stabilizing region of (k_i, k_d) for: (a) $-(1/k) < k_p < 1/k$. (b) $k_p = 1/k$. (c) $1/k < k_p < k_u$.

B. Open-Loop Unstable Plant

In this case, $T < 0$ in (2.1). Furthermore, let us assume that $k > 0$ and $L > 0$.

Theorem 2.2: A necessary and sufficient condition for the existence of a stabilizing PID controller for the open-loop unstable plant (2.1) is $|T/L| > 0.5$. If this condition is satisfied, then the range of k_p values for which a given open-loop unstable plant, with transfer function $G(s)$ as in (2.1), can be stabilized using a PID controller is given by

$$\frac{1}{k}\left[\frac{T}{L}\,\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right] < k_p < -\frac{1}{k} \tag{2.4}$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\,\alpha \tag{2.5}$$

in the interval $(0, \pi)$. In the special case of $|T/L| = 1$, we have $\alpha_1 = \pi/2$. For k_p values outside this range, there are no stabilizing PID controllers. Moreover, the complete stabilizing region is characterized by: (see Fig. 4)

For each $k_p \in (k_l := 1/k[(T/L)\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)], -(1/k))$, the cross section of the stabilizing region in the (k_i, k_d) space is the quadrilateral Q.

The parameters m_j, b_j and w_j, $j = 1, 2$ necessary for determining the boundary of Q are as defined in the statement of Theorem 2.1.

III. Preliminary Results for Analyzing Systems With Time Delays

Many problems in process control engineering involve time delays. These time delays lead to dynamic models with characteristic equations of the form

$$\delta(s) = d(s) + e^{-sT_1} n_1(s) + e^{-sT_2} n_2(s) + \cdots + e^{-sT_m} n_m(s) \tag{3.1}$$

where $d(s)$, $n_i(s)$ for $i = 1, 2, \ldots, m$, are polynomials with real coefficients. Characteristic equations of this form are

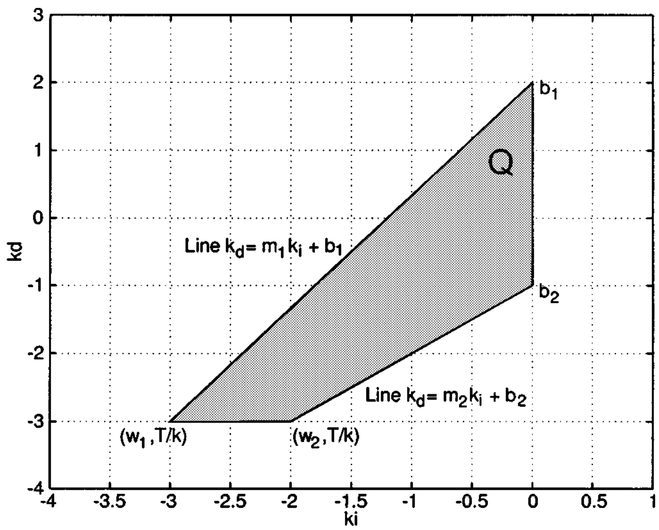

Fig. 4. The stabilizing region of (k_i, k_d) for $k_l < k_p < -(1/k)$.

known as quasipolynomials. It can be shown that the so called Hermite–Biehler Theorem for Hurwitz polynomials [6], [7], does not carry over to arbitrary functions $f(s)$ of the complex variable s. Pontryagin [4] studied entire functions of the form $P(s, e^s)$, where $P(s, t)$ is a polynomial in two variables and is called a quasipolynomial. Based on Pontryagin's results, a suitable extension of the Hermite–Biehler Theorem can be developed (see [8], [7] and the references therein) to study the stability of certain classes of quasipolynomials characterized as follows. If in (3.1) we make the assumptions

A1) $\deg[d(s)] = n$ and $\deg[n_i(s)] \le n$ for $i = 1, 2, \dots, m$;
A2) $0 < T_1 < T_2 < \cdots < T_m$,

then instead of (3.1) we can consider the quasipolynomial

$$\begin{aligned}\delta^*(s) &= e^{sT_m}\delta(s)\\ &= e^{sT_m}d(s) + e^{s(T_m-T_1)}n_1(s)\\ &\quad + e^{s(T_m-T_2)}n_2(s) + \cdots + n_m(s). \qquad (3.2)\end{aligned}$$

Since e^{sT_m} does not have any finite zeros, the zeros of $\delta(s)$ are identical to those of $\delta^*(s)$. The quasipolynomial $\delta^*(s)$, however, has a principal term [8], i.e., the coefficient of the term containing the highest powers of s and e^s is nonzero. It then follows that this quasipolynomial is either of the *delay* or of the *neutral* type [9]. This being the case, the stability of the system with characteristic equation (3.1) is equivalent to the condition that all the zeros of $\delta^*(s)$ be in the open left-half plane. We will say equivalently that $\delta^*(s)$ is Hurwitz or is stable. The following theorem gives necessary and sufficient conditions for the stability of $\delta^*(s)$ [8].

Theorem 3.1: Let $\delta^*(s)$ be given by (3.2), and write

$$\delta^*(jw) = \delta_r(w) + j\delta_i(w)$$

where $\delta_r(w)$ and $\delta_i(w)$ represent respectively the real and imaginary parts of $\delta^*(jw)$. Under assumptions A1) and A2), $\delta^*(s)$ is stable if and only if

1) $\delta_r(w)$ and $\delta_i(w)$ have only simple real roots and these interlace;
2) $\delta_i'(w_o)\delta_r(w_o) - \delta_i(w_o)\delta_r'(w_o) > 0$, for some w_o in $(-\infty, \infty)$;

where $\delta_r'(w)$ and $\delta_i'(w)$ denote the first derivative with respect to w of $\delta_r(w)$ and $\delta_i(w)$, respectively.

In the rest of this paper, we will be making use of this theorem to provide a solution to the PID stabilization problem for first-order plants with dead time. A crucial step in applying the above theorem to check stability is to ensure that $\delta_r(w)$ and $\delta_i(w)$ have only *real* roots. Such a property can be ensured by using the following result, also due to Pontryagin [8].

Theorem 3.2: Let M and N denote the highest powers of s and e^s respectively in $\delta^*(s)$. Let η be an appropriate constant such that the coefficients of terms of highest degree in $\delta_r(w)$ and $\delta_i(w)$ do not vanish at $w = \eta$. Then for the equations $\delta_r(w) = 0$ or $\delta_i(w) = 0$ to have only real roots, it is necessary and sufficient that in the intervals

$$-2l\pi + \eta \le w \le 2l\pi + \eta, \qquad l = 1, 2, 3, \dots$$

$\delta_r(w)$ or $\delta_i(w)$ have exactly $4lN + M$ real roots starting with a sufficiently large l.

IV. Stabilization Using a PID Controller

We first analyze the system shown in Fig. 2 without the time delay, i.e., $L = 0$. In this case, the closed-loop characteristic equation of the system is given by

$$\delta(s) = (T + kk_d)s^2 + (1 + kk_p)s + kk_i.$$

Since this is a second-order polynomial, closed loop stability is equivalent to all the coefficients having the same sign. Assuming that the steady-state gain k of the plant is positive these conditions are

$$k_p > -\frac{1}{k}, \quad k_i > 0 \quad \text{and} \quad k_d > -\frac{T}{k} \qquad (4.1)$$

or

$$k_p < -\frac{1}{k}, \quad k_i < 0 \quad \text{and} \quad k_d < -\frac{T}{k}. \qquad (4.2)$$

Now, a minimal requirement for any control design is that the delay-free closed-loop system be stable. Consequently, it will be henceforth assumed in this paper that the PID gains used to stabilize the plant with delay always satisfy one of the conditions (4.1) or (4.2).

Next, consider the case where the time delay of the plant model is different from zero. The closed-loop characteristic equation of the system is then

$$\delta(s) = (kk_i + kk_p s + kk_d s^2)e^{-Ls} + (1 + Ts)s.$$

Due to the presence of the exponential term, the number of roots of the above quasipolynomial may be infinite. This makes the problem of analyzing the stability of the closed-loop system a difficult one. However, we can make use of Theorems 3.1 and 3.2 to solve the stability problem and find the set of stabilizing PID controllers.

We start by rewriting the quasipolynomial $\delta(s)$ as

$$\delta^*(s) = e^{Ls}\delta(s) = kk_i + kk_p s + kk_d s^2 + (1 + Ts)se^{Ls}.$$

Substituting $s = jw$, we have

$$\delta^*(jw) = \delta_r(w) + j\delta_i(w)$$

where

$$\delta_r(w) = kk_i - kk_d w^2 - w\,\sin(Lw) - Tw^2\,\cos(Lw)$$
$$\delta_i(w) = w[kk_p + \cos(Lw) - Tw\,\sin(Lw)].$$

From the expressions for $\delta_r(w)$ and $\delta_i(w)$, it is clear that the controller parameter k_p only affects the imaginary part of $\delta^*(jw)$ whereas the parameters k_i and k_d affect the real part of $\delta^*(jw)$. Moreover, these three controller parameters appear affinely in $\delta_r(w)$ and $\delta_i(w)$. These facts are exploited in applying Theorems 3.1 and 3.2 to determine the range of stabilizing PID gains.

We now consider two different cases.

A. Open-Loop Stable Plant ($T > 0$)

The proof of the main result Theorem 2.1 makes use of several lemmas. These are presented next.

Lemma 4.1: The imaginary part of $\delta^*(jw)$ has only simple real roots if and only if

$$-\frac{1}{k} < k_p < \frac{1}{k}\left[\frac{T}{L}\alpha_1\,\sin(\alpha_1) - \cos(\alpha_1)\right] \tag{4.3}$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\,\alpha$$

in the interval $(0, \pi)$.

Proof: With the change of variables $z = Lw$ the real and imaginary parts of $\delta^*(jw)$ can be expressed as

$$\delta_r(z) = kk_i - \frac{kk_d}{L^2}\,z^2 - \frac{1}{L}\,z\,\sin(z) - \frac{T}{L^2}\,z^2\,\cos(z) \tag{4.4}$$
$$\delta_i(z) = \frac{z}{L}\left[kk_p + \cos(z) - \frac{T}{L}\,z\,\sin(z)\right]. \tag{4.5}$$

From (4.5), we can compute the roots of the imaginary part, i.e., $\delta_i(z) = 0$. This gives us the following equation:

$$\frac{z}{L}\left[kk_p + \cos(z) - \frac{T}{L}\,z\,\sin(z)\right] = 0.$$

Then either

$$z = 0 \quad \text{or}$$
$$kk_p + \cos(z) - \frac{T}{L}\,z\,\sin(z) = 0. \tag{4.6}$$

From this it is clear that one root of the imaginary part is $z_o = 0$. The other roots are difficult to find since we need to solve (4.6) analytically. However, we can plot the terms involved in equation (4.6) and graphically examine the nature of the solution. Let us denote the positive-real roots of (4.6) by z_j, $j = 1, 2, \ldots$, arranged in increasing order of magnitude. There are now four different cases to consider.

Case 1) $k_p < -(1/k)$. In this case, we sketch $(kk_p + \cos(z))/\sin(z)$ and $(T/L)z$ to obtain the plots shown in Fig. 5.

Case 2) $-(1/k) < k_p < 1/k$. In this case, we graph $(kk_p + \cos(z))/\sin(z)$ and $(T/L)z$ to obtain the plots shown in Fig. 6.

Case 3) $k_p = 1/k$. In this case, we sketch $kk_p + \cos(z)$ and $(T/L)z\,\sin(z)$ to obtain the plots shown in Fig. 7.

Case 4) $1/k < k_p$. In this case, we sketch $(kk_p + \cos(z))/\sin(z)$ and $(T/L)z$ to obtain the plots shown in Fig. 8(a) and (b). The plot in Fig. 8(a) corresponds to the case where $1/k < k_p < k_u$, and k_u is the largest number so that the plot of $(kk_p + \cos(z))/\sin(z)$ intersects the line $(T/L)z$ twice in the interval $(0, \pi)$. The plot in Fig. 8(b) corresponds to the case where $k_p \geq k_u$ and the plot of $(kk_p + \cos(z))/\sin(z)$ does not intersect the line $(T/L)z$ twice in the interval $(0, \pi)$.

Let us now use Theorem 3.2 to check if $\delta_i(z)$ has only real roots. Substituting $s_1 = Ls$ in the expression for $\delta^*(s)$, we see that for the new quasipolynomial in s_1, $M = 2$ and $N = 1$. Next we choose $\eta = \pi/4$ to satisfy the requirement that $\sin(\eta) \neq 0$. Now from Figs. 6–8(a), we see that in each of these cases, i.e., for $-(1/k) < k_p < k_u$, $\delta_i(z)$ has three real roots in the interval $[0, 2\pi - (\pi/4)] = [0, 7\pi/4]$, including a root at the origin. Since $\delta_i(z)$ is an odd function of z, it follows that in the interval $[-(7\pi/4), 7\pi/4]$, $\delta_i(z)$ will have 5 real roots. Also observe from Figs. 6–8(a) that $\delta_i(z)$ has a real root in the interval $(7\pi/4, 9\pi/4]$. Thus $\delta_i(z)$ has $4N + M = 6$ real roots in the interval $[-2\pi + (\pi/4), 2\pi + (\pi/4)]$. Moreover, it is clear from Figs. 6–8(a) that $\delta_i(z)$ has two real roots in each of the intervals $[2l\pi + (\pi/4), 2(l+1)\pi + (\pi/4)]$ and $[-2(l+1)\pi + (\pi/4), -2l\pi + (\pi/4)]$ for $l = 1, 2, \ldots$ Hence, it follows that $\delta_i(z)$ has exactly $4lN + M$ real roots in $[-2l\pi + (\pi/4), 2l\pi + (\pi/4)]$ for $-(1/k) < k_p < k_u$. Hence, from Theorem 3.2, we conclude that for $-(1/k) < k_p < k_u$, $\delta_i(z)$ has only real roots. Also note that the cases $k_p < -(1/k)$ and $k_p \geq k_u$ corresponding to Figs. 5 and 8(b), respectively, do not merit any further consideration since using Theorem 3.2, one can easily argue that in these cases, all the roots of $\delta_i(z)$ will not be real, thereby ruling out closed-loop stability. It only remains to determine the upper bound k_u on the allowable value of k_p. From the definition of k_u, it follows that if $k_p = k_u$ the plot of $(kk_p + \cos(z))/\sin(z)$ intersects the line $(T/L)z$ only once in the interval $(0, \pi)$. Let us denote by α_1 the value of z for which this intersection occurs. Then, we know that for $z = \alpha_1 \in (0, \pi)$ we have

$$\frac{kk_u + \cos(\alpha_1)}{\sin(\alpha_1)} = \frac{T}{L}\,\alpha_1. \tag{4.7}$$

Moreover, at $z = \alpha_1$, the line $(T/L)z$ is tangent to the plot of $(kk_u + \cos(z))/\sin(z)$. Thus

$$\frac{d}{dz}\left[\frac{kk_u + \cos(z)}{\sin(z)}\right]_{z=\alpha_1} = \frac{T}{L}$$
$$\Rightarrow 1 + kk_u\,\cos(\alpha_1) = -\frac{T}{L}\,\sin^2(\alpha_1). \tag{4.8}$$

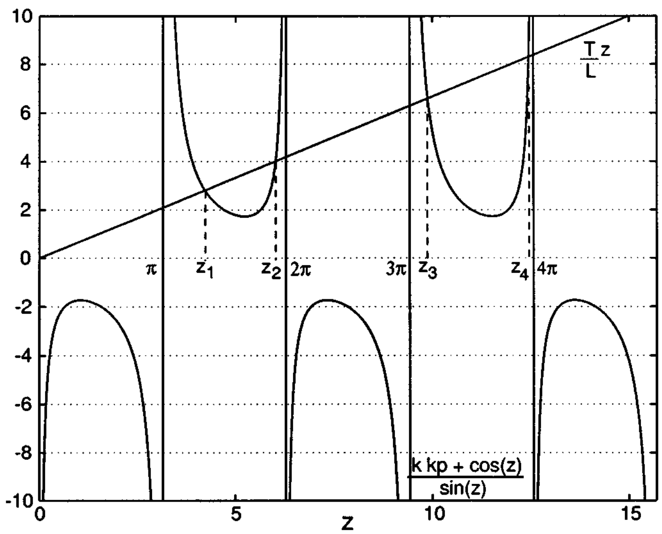

Fig. 5. Plot of the terms involved in equation (4.6) for $k_p < -(1/k)$.

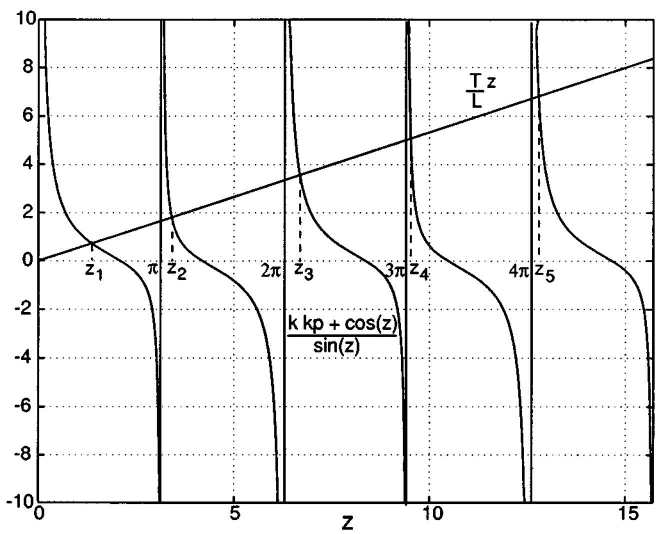

Fig. 6. Plot of the terms involved in equation (4.6) for $-(1/k) < k_p < 1/k$.

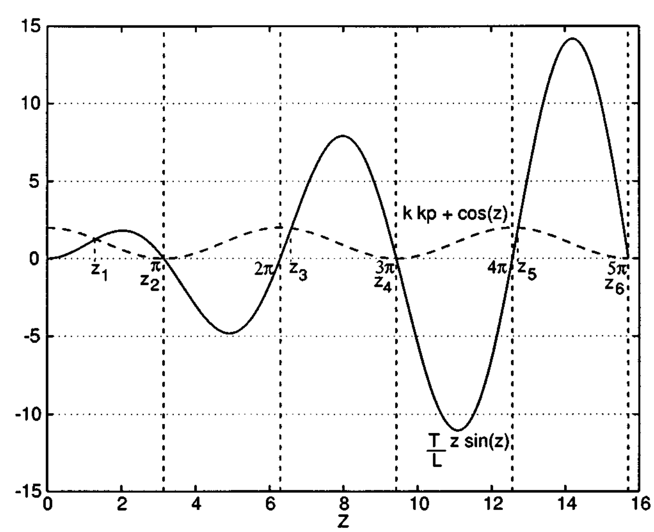

Fig. 7. Plot of the terms involved in equation (4.6) for $k_p = 1/k$.

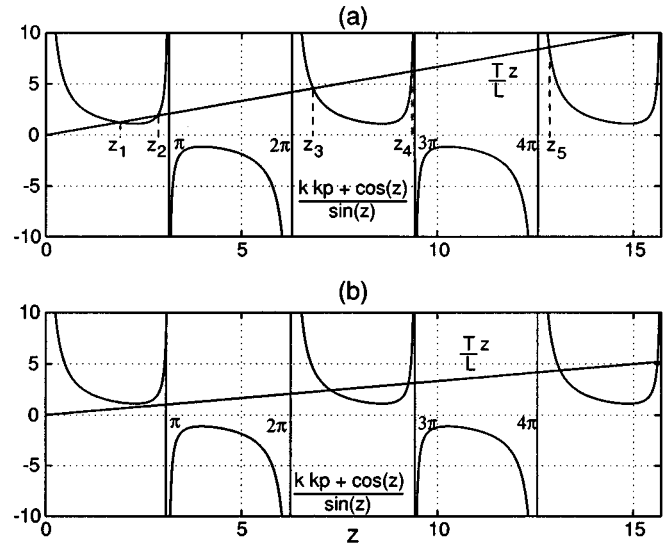

Fig. 8. Plot of the terms involved in equation (4.6) for $1/k < k_p$.

Eliminating kk_u between equations (4.8) and (4.7) we conclude that $\alpha_1 \in (0, \pi)$ can be obtained as a solution of the following equation:

$$\tan(\alpha_1) = -\frac{T}{T+L}\,\alpha_1.$$

Once α_1 is determined the parameter k_u can be obtained using (4.7)

$$k_u = \frac{1}{k}\left[\frac{T}{L}\,\alpha_1\,\sin(\alpha_1) - \cos(\alpha_1)\right].$$

This completes the proof of the lemma. ♣

Now, from (4.4), for $z \neq 0$, the real part $\delta_r(z)$ can be rewritten as

$$\delta_r(z) = \frac{k}{L^2}\,z^2[-k_d + m(z)k_i + b(z)] \tag{4.9}$$

where

$$m(z) \triangleq \frac{L^2}{z^2} \tag{4.10}$$

$$b(z) \triangleq -\frac{L}{kz}\left[\sin(z) + \frac{T}{L}\,z\,\cos(z)\right]. \tag{4.11}$$

Lemma 4.2: For every value of k_p in the range given by (4.3), the necessary and sufficient conditions on k_i and k_d for the roots of $\delta_r(z)$ and $\delta_i(z)$ to interlace are the following infinite set of inequalities:

$$\begin{aligned} &k_i > 0\\ &(-1)^j k_d < (-1)^j m_j k_i + (-1)^j b_j,\\ &\qquad j = 1, 2, 3, \ldots \end{aligned} \tag{4.12}$$

where the parameters m_j and b_j for $j = 1, 2, 3, \ldots$ are given by

$$m_j \triangleq m(z_j) \tag{4.13}$$

$$b_j \triangleq b(z_j). \tag{4.14}$$

Proof: From Condition 1 of Theorem 3.1, the roots of $\delta_r(z)$ and $\delta_i(z)$ have to interlace in order for the quasipolynomial $\delta^*(s)$ to be stable. Thus, we evaluate $\delta_r(z)$ at the roots of the imaginary part $\delta_i(z)$. For $z_o = 0$, using (4.4) we obtain

$$\delta_r(z_o) = kk_i. \tag{4.15}$$

For z_j, where $j = 1, 2, 3, \ldots$, using (4.9) we obtain

$$\delta_r(z_j) = \frac{k}{L^2} z_j^2[-k_d + m(z_j)k_i + b(z_j)]. \tag{4.16}$$

Interlacing of the roots of $\delta_r(z)$ and $\delta_i(z)$ is equivalent to $\delta_r(z_o) > 0$ [since Lemma 4.1 implies that k_p is necessarily greater than $-(1/k)$, which in view of the stability requirements (4.1) for the delay-free case implies that $k_i > 0$], $\delta_r(z_1) < 0$, $\delta_r(z_2) > 0$, $\delta_r(z_3) < 0$, and so on. Using this fact and equations (4.15) and (4.16) we obtain

$$\begin{aligned} &\delta_r(z_o) > 0 \Rightarrow k_i > 0 \\ &(-1)^j\delta_r(z_j) > 0 \Rightarrow (-1)^j k_d < (-1)^j m_j k_i + (-1)^j b_j, \\ &\qquad j = 1, 2, 3, \ldots. \end{aligned}$$

Thus, intersecting all these regions in the k_i–k_d space, we obtain the set of (k_i, k_d) values for which the roots of $\delta_r(z)$ and $\delta_i(z)$ interlace for a given fixed value of k_p. Notice that all these regions are half planes with their boundaries being lines with positive slopes m_j. This completes the proof of the lemma. ♣

As pointed out in the proof of Lemma 4.2, the inequalities given by (4.12) represent half planes in the space of k_i and k_d. Their boundaries are given by lines with the following equations:

$$k_d = m_j k_i + b_j \qquad \text{for } j = 1, 2, 3 \ldots.$$

The focus of the remainder of this subsection will be to show that this intersection is nonempty. We will also determine the intersection of this *countably infinite number* of half planes in a computationally tractable way. To this end, let us denote by v_j the k_i-coordinate of the intersection of the line $k_d = m_j k_i + b_j$, $j = 1, 2, 3 \cdots$, with the line $k_d = -(T/k)$. From (4.13) and (4.14), it is not difficult to show that

$$v_j = \frac{z_j}{kL}\left[\sin(z_j) + \frac{T}{L} z_j(\cos(z_j) - 1)\right]. \tag{4.17}$$

In a similar fashion, let us now denote by w_j the k_i-coordinate of the intersection of the line $k_d = m_j k_i + b_j$, $j = 1, 2, 3 \ldots$, with the line $k_d = T/k$. Using (4.13) and (4.14) it can be once again shown that

$$w_j = \frac{z_j}{kL}\left[\sin(z_j) + \frac{T}{L} z_j(\cos(z_j) + 1)\right]. \tag{4.18}$$

We now state three important technical lemmas that will allow us to develop an algorithm for solving the PID stabilization problem of an open-loop stable plant ($T > 0$). These lemmas show the behavior of the parameters b_j, v_j and w_j, $j = 1, 2, 3, \ldots$, for different values of the parameter k_p inside the range proposed by Lemma 4.1. The proofs of these lemmas are long and technical and are therefore presented in Appendix A.

Lemma 4.3: If $-(1/k) < k_p < 1/k$ then

i) $b_j < b_{j+2} < -(T/k)$ for odd values of j;
ii) $b_j > T/k$ and $b_j \to T/k$ as $j \to \infty$ for even values of j;
iii) $0 < v_j < v_{j+2}$ for odd values of j.

Lemma 4.4: If $k_p = 1/k$ then

i) $b_j = -(T/k)$ for odd values of j;and
ii) $b_j = T/k$ for even values of j.

Lemma 4.5: If $1/k < k_p < 1/k[(T/L)\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)]$ where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\,\alpha$$

in the interval $(0, \pi)$, then

i) $b_j > b_{j+2} > -(T/k)$ for odd values of j;
ii) $b_j < b_{j+2} < T/k$ for even values of j;
iii) $w_j > w_{j+2} > 0$ for even values of j;
iv) $b_1 < b_2$, $w_1 > w_2$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1: To ensure the stability of the quasipolynomial $\delta^*(s)$ we need to check the two conditions given in Theorem 3.1.

Step 1) We first check Condition 2 of Theorem 3.1

$$E(w_o) = \delta_i'(w_o)\delta_r(w_o) - \delta_i(w_o)\delta_r'(w_o) > 0$$

for some w_o in $(-\infty, \infty)$. Let us take $w_o = z_o = 0$. Thus, $\delta_i(z_o) = 0$ and $\delta_r(z_o) = kk_i$. We also have

$$\begin{aligned} \delta_i'(z) &= \frac{kk_p}{L} + \left(\frac{1}{L} - \frac{T}{L^2} z^2\right)\cos(z) \\ &\quad - \left(\frac{1}{L} z + \frac{2T}{L^2} z\right)\sin(z) \\ \Rightarrow E(z_o) &= \left(\frac{kk_p + 1}{L}\right)(kk_i). \end{aligned}$$

Recall that $k > 0$ and $L > 0$. Thus, if we pick

$$k_i > 0 \quad \text{and} \quad k_p > -\frac{1}{k} \tag{4.19}$$

or

$$k_i < 0 \quad \text{and} \quad k_p < -\frac{1}{k}$$

we have $E(z_o) > 0$. Notice that from these conditions one can safely discard $k_p = -(1/k)$ from the set of k_p values for which a stabilizing PID controller can be found.

Step 2) Next, we check condition 1 of Theorem 3.1, i.e., $\delta_r(z)$ and $\delta_i(z)$ have only simple real roots and these interlace. From Lemma 4.1, we know that the roots of $\delta_i(z)$ are all real if and only if the parameter k_p lies inside the range $(-(1/k), 1/k[(T/L)\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)])$, where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\,\alpha$$

in the interval $(0, \pi)$. Now, from the proof of Lemma 4.2, we see that interlacing of the roots of $\delta_r(z)$ and $\delta_i(z)$ leads to the following set of inequalities

$$k_i > 0$$
$$(-1)^j k_d < (-1)^j m_j k_i + (-1)^j b_j, \qquad j = 1, 2, 3, \ldots.$$

We now show that for $-(1/k) < k_p < k_u$, where $k_u = 1/k[(T/L)\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)]$, all these regions do have a nonempty intersection. Notice first that the slopes m_j of the boundary lines of these regions decrease with z_j. Moreover, in the limit we have:

$$\lim_{j\to\infty} m_j = 0.$$

Using this fact we have the following observations.

1) When $-(1/k) < k_p < 1/k$, the intersection is given by the trapezoid T sketched in Fig. 3(a). This region can be found using the properties stated in Lemma 4.3.
2) When $k_p = 1/k$, the intersection is given by the triangle Δ sketched in Fig. 3(b). This region can be found using the properties stated in Lemma 4.4.
3) When $1/k < k_p < k_u$, the intersection is given by the quadrilateral Q sketched in Fig. 3(c). This region can be found using the properties stated in Lemma 4.5.

Now, for values of k_p in $(-(1/k), k_u)$, the interlacing property and the fact that the roots of $\delta_i(z)$ are all real can be used in Theorem 3.2 to guarantee that $\delta_r(z)$ also has only real roots. Thus, for values of k_p inside this range there is a solution to the PID stabilization problem for a first-order open-loop stable plant with time delay. For values of k_p outside this range the aforementioned problem does not have a solution. This completes the proof of the theorem. ♣

Remark 4.1: For $L = 0$, i.e., no time delay, one can solve (2.3) analytically to obtain $\alpha_1 = 2.0288$. Using this value, the upper bound in (2.2) evaluates out to ∞, which is consistent with the condition imposed on k_p by (4.1), for one of the scenarios arising in the delay-free case. Also, by plotting the graphs of $\tan(\alpha)$ and $-(T/(T+L))\alpha$ versus α, it is easy to see that as T/L decreases, the intersection α_1 approaches π. By substituting for T/L from (2.3) into the upper bound in (2.2) and differentiating with respect to α, it can be shown that as α increases from 2.0288 and approaches π, the upper bound in (2.2) monotonically decreases to $1/k$ (see also Fig. 9). This shows that as L/T increases, the range of k_p values shrinks, and this is consistent with the empirical observation in [2, p. 16].

In view of Theorem 2.1, we now propose an algorithm to determine the set of stabilizing parameters for the plant (2.1) with $T > 0$.

Algorithm for Determining Stabilizing PID Parameters

- **Step 1**: Pick a k_p in the range dictated by Theorem 2.1.
- **Step 2**: Find the roots z_1 and z_2 of equation (4.6).
- **Step 3**: Compute the parameters m_j and b_j, $j = 1, 2$ associated with the previously found z_j by using equations (4.13) and (4.14).
- **Step 5**: Determine the stabilizing region in the k_i–k_d space using Fig. 3.
- **Step 6**: Go to Step 1.

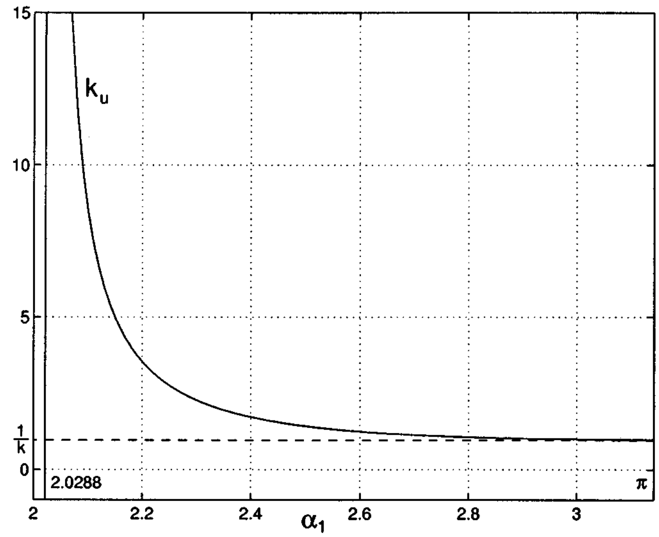

Fig. 9. Plot of the upper bound in (2.2) as a function of α_1.

B. Open-Loop Unstable Plant ($T < 0$)

The proof of the main result Theorem 2.2 follows similar steps as in the case of Theorem 2.1. First, the following lemma which is the unstable plant counterpart of Lemma 4.1 shows that the range of k_p values for which PID stabilization is possible can be determined exactly.

Lemma 4.6: For $|T/L| > 0.5$, the imaginary part of $\delta^*(jw)$ has only simple real roots if and only if

$$\frac{1}{k}\left[\frac{T}{L}\alpha_1 \sin(\alpha_1) - \cos(\alpha_1)\right] < k_p < -\frac{1}{k} \tag{4.20}$$

where α_1 is the solution of the equation

$$\tan(\alpha) = -\frac{T}{T+L}\alpha$$

in the interval $(0, \pi)$. In the special case of $|T/L| = 1$, we have $\alpha_1 = \pi/2$. For $|T/L| \le 0.5$, the roots of the imaginary part of $\delta^*(jw)$ are not all real.

Proof: The proof follows along the same lines as that of Lemma 4.1 with appropriate modifications which are fairly obvious. The only nonobvious change is that in treating Case 1, i.e., $k_p < -(1/k)$, one has to make use of the following lemma to establish that for $|T/L| \le 0.5$, the roots of the imaginary part of $\delta^*(j\omega)$ are not all real, regardless of the value of k_p in this range. By Theorem 3.1, this implies instability. ♣

Lemma 4.7: If $-0.5 \le T/L < 0$, then the curves $(kk_p + \cos(z))/\sin(z)$ and $(T/L)z$ do not intersect in the interval $(0, \pi)$ regardless of the value of k_p in $(-\infty, -(1/k))$.

Proof: The proof can be found in Appendix B. ♣

Proof of Theorem 2.2: The proof is similar to that of Theorem 2.1 and is based on developing the appropriate counterparts of Lemmas 4.2 and 4.5. Due to space limitations, we omit the details and refer the interested reader to [10]. ♣

A similar algorithm to the one presented in the previous subsection can now be developed to solve the PID stabilization problem of an open-loop unstable plant. One only needs to sweep the parameter k_p over the interval proposed by Theorem 2.2 and use Fig. 4 to find the stabilizing region of (k_i, k_d) values at each admissible value of k_p.

V. Concluding Remarks

In this paper we have presented a procedure to determine the complete set of stabilizing PID controllers for a given first-order plant with dead-time. The procedure is based on first determining the range of proportional gain values for which a solution to the PID stabilization problem exists. Then, it is shown that for a fixed proportional gain value inside this range, the stabilizing integral and derivative gain values lie inside a region with known shape and boundaries. Furthermore, it has been demonstrated that this region can be characterized in a computationally tractable manner. By sweeping over the entire range of allowable proportional gain values and determining the stabilizing regions in the space of the integral and derivative gains, the complete set of stabilizing PID controllers can be determined. The results presented here are based on an extension of the Hermite–Biehler Theorem to quasipolynomials. It is our belief that the results of this paper will form the basis for developing computationally efficient tools for PID controller design and analysis.

Appendix A
Proof of Lemmas 4.3, 4.4, and 4.5

We begin by making the following observations which follow from the proof of Lemma 4.1.

Remark A.1: Notice from Figs. 6–8(a), for $k_p \in (-(1/k), k_u)$, the odd roots of (4.6), i.e., z_j where $j = 1, 3, 5, \ldots$ are getting closer to $(j-1)\pi$ as j increases. So in the limit for odd values of j we have:

$$\lim_{j\to\infty} \cos(z_j) = 1.$$

Moreover, since the cosine function is monotonically decreasing between $(j-1)\pi$ and $j\pi$ for odd values of j, in view of the previous observation we have:

$$\cos(z_1) < \cos(z_3) < \cos(z_5) < \cdots$$

Remark A.2: From Figs. 6 and 8(a) notice that for $k_p \in (-(1/k), 1/k) \cup (1/k, k_u)$, the even roots of (4.6), i.e., z_j where $j = 2, 4, 6, \ldots$ are getting closer to $(j-1)\pi$ as j increases. So in the limit for even values of j we have:

$$\lim_{j\to\infty} \cos(z_j) = -1.$$

We also notice in Fig. 6 that these roots approach $(j-1)\pi$ from the right whereas in Fig. 8(a) they approach $(j-1)\pi$ from the left. Now, since the cosine function is monotonically decreasing between $(j-2)\pi$ and $(j-1)\pi$ $(j = 2, 4, 6, \ldots)$ and is monotonically increasing between $(j-1)\pi$ and $j\pi$ $(j = 2, 4, 6, \ldots)$, we have:

$$\cos(z_2) > \cos(z_4) > \cos(z_6) > \cdots$$

for $k_p \in (-(1/k), 1/k) \cup (1/k, k_u)$. In the particular case of Fig. 7, i.e., $k_p = 1/k$, we notice that $\cos(z_2) = \cos(z_4) = \cos(z_6) = \cdots = -1$.

Before proving Lemmas 4.3, 4.4, and 4.5, we first state and prove the following technical lemmas that will simplify the subsequent analysis.

Lemma A.1: Consider the function $E_1\colon \mathcal{Z}^+ \times \mathcal{Z}^+ \to \mathcal{R}$ defined by

$$E_1(m, n) \triangleq b_m - b_n$$

where m and n are natural numbers and b_j, $j = m, n$, are as defined in (4.14). Then, for $z_m, z_n \neq l\pi$, $l = 0, 1, 2, \ldots$, $E_1(m, n)$ can be equivalently expressed as

$$E_1(m, n) = \frac{L^2}{kT}\,\frac{[1-(kk_p)^2][\cos(z_m)-\cos(z_n)]}{z_m z_n \sin(z_m)\sin(z_n)}.$$

Proof: We will first show that for $z_j \neq l\pi$, $j = 1, 2, 3, \ldots$, the following identity holds:

$$\sin(z_j) + \frac{T}{L}\, z_j \cos(z_j) = \frac{1 + kk_p \cos(z_j)}{\sin(z_j)}. \tag{A.1}$$

Now for $z_j \neq l\pi$, from (4.6), we obtain

$$\begin{aligned}\sin(z_j) &+ \frac{T}{L}\, z_j \cos(z_j)\\ &= \sin(z_j) + \left[\frac{kk_p + \cos(z_j)}{\sin(z_j)}\right]\cos(z_j)\\ &= \frac{1 + kk_p\cos(z_j)}{\sin(z_j)}.\end{aligned}$$

Now from (4.14) we can rewrite $E_1(m, n)$ as follows:

$$\begin{aligned}&E_1(m, n)\\ &\quad = -\frac{L}{kz_m}\left[\sin(z_m) + \frac{T}{L}\, z_m\cos(z_m)\right]\\ &\qquad + \frac{L}{kz_n}\left[\sin(z_n) + \frac{T}{L}\, z_n \cos(z_n)\right]\\ &\Rightarrow -\frac{k}{L}\, E_1(m, n)\\ &\quad = \frac{1 + kk_p\cos(z_m)}{z_m \sin(z_m)} - \frac{1 + kk_p\cos(z_n)}{z_n\sin(z_n)} \quad \text{[using (A.1)]}\\ &\quad = \frac{z_n\sin(z_n)[1+kk_p\cos(z_m)] - z_m\sin(z_m)[1+kk_p\cos(z_n)]}{z_m z_n \sin(z_m)\sin(z_n)}.\end{aligned}$$

Now, since z_j, $j = 1, 2, 3, \ldots$, satisfy (4.6), we can rewrite the previous expression as shown in the equation at the bottom of the next page. Thus, we finally obtain

$$E_1(m, n) = \frac{L^2}{kT}\,\frac{[1-(kk_p)^2][\cos(z_m)-\cos(z_n)]}{z_m z_n\sin(z_m)\sin(z_n)}.$$

♣

Before stating the next lemma, we introduce the standard signum function $\operatorname{sgn}\colon \mathcal{R} \to \{-1, 0, 1\}$ defined by

$$\operatorname{sgn}[x] = \begin{cases} -1 & \text{if } x < 0\\ 0 & \text{if } x = 0\\ 1 & \text{if } x > 0.\end{cases}$$

Lemma A.2: Consider the function E_2: $\mathcal{Z}^+ \times \mathcal{Z}^+ \rightarrow R$ defined by

$$E_2(m, n) \triangleq v_m - v_n$$

where m and n are natural numbers and v_j, $j = m, n$, are as defined in (4.17). If $k_p \neq 1/k$ and $z_m, z_n \neq l\pi, l = 1, 2, 3, \ldots$, then

$$\text{sgn}[E_2(m, n)] = \text{sgn}[T] \cdot \text{sgn}[\cos(z_m) - \cos(z_n)].$$

Proof: First, since z_j, $j = 1, 2, 3, \ldots$, satisfies (4.6), we can rewrite v_j as follows:

$$v_j = \frac{z_j}{kL}\left[\sin(z_j) + \frac{kk_p + \cos(z_j)}{\sin(z_j)}(\cos(z_j) - 1)\right]$$
$$\Rightarrow v_j = \frac{z_j}{kL}\,\frac{(1 - kk_p)[1 - \cos(z_j)]}{\sin(z_j)}. \quad \text{(A.2)}$$

Now, using (A.2) the function $E_2(m, n)$ can be equivalently expressed as

$$E_2(m, n) = \frac{z_m}{kL}\,\frac{(1 - kk_p)[1 - \cos(z_m)]}{\sin(z_m)} - \frac{z_n}{kL}\,\frac{(1 - kk_p)[1 - \cos(z_n)]}{\sin(z_n)}$$
$$\Rightarrow \frac{kL}{1 - kk_p}\,E_2(m, n) = z_m\frac{[1-\cos(z_m)]}{\sin(z_m)} - z_n\frac{[1-\cos(z_n)]}{\sin(z_n)}.$$

Once more we use the fact that z_j, $j = 1, 2, 3, \ldots$, satisfies (4.6):

$$\frac{kL}{1 - kk_p}\,E_2(m, n) = \frac{[kk_p + \cos(z_m)][1 - \cos(z_m)]}{\frac{T}{L}\sin^2(z_m)} - \frac{[kk_p + \cos(z_n)]\,[1 - \cos(z_n)]}{\frac{T}{L}\sin^2(z_n)}$$
$$\Rightarrow \frac{kT}{1 - kk_p}\,E_2(m, n) = \frac{kk_p + \cos(z_m)}{1 + \cos(z_m)} - \frac{kk_p + \cos(z_n)}{1 + \cos(z_n)}$$
$$[\text{since } \sin^2(x) = 1 - \cos^2(x)]$$
$$= \frac{(1 - kk_p)[\cos(z_m) - \cos(z_n)]}{[1 + \cos(z_m)][1 + \cos(z_n)]}.$$

Thus, the function $E_2(m, n)$ is given by

$$E_2(m, n) = \frac{(1 - kk_p)^2[\cos(z_m) - \cos(z_n)]}{kT[1 + \cos(z_m)]\,[1 + \cos(z_n)]}.$$

Now, since $k_p \neq 1/k$, then $(1 - kk_p)^2 > 0$. Also, since z_m, $z_n \neq (2l - 1)\pi, l = 1, 2, 3, \ldots$, then $1 + \cos(z_m) > 0$ and $1 + \cos(z_n) > 0$. Thus, from the previous expression for $E_2(m, n)$ it is clear that

$$\text{sgn}[E_2(m, n)] = \text{sgn}[T] \cdot \text{sgn}[\cos(z_m) - \cos(z_n)].$$

This completes the proof of the lemma. ♣

Lemma A.3: Consider the function E_3: $\mathcal{Z}^+ \times \mathcal{Z}^+ \rightarrow R$ defined by

$$E_3(m, n) \triangleq w_m - w_n$$

where m and n are natural numbers and w_j, $j = m, n$, are as defined in (4.18). If $k_p \neq -1/k$ and z_m, $z_n \neq l\pi$, $l = 1, 2, 3, \ldots$, then

$$\text{sgn}[E_3(m, n)] = \text{sgn}[T] \cdot \text{sgn}[\cos(z_m) - \cos(z_n)].$$

Proof: As in the previous proof, we use the fact that z_j, $j = 1, 2, 3, \ldots$, satisfies (4.6). Thus, w_j can be rewritten as follows:

$$w_j = \frac{z_j}{kL}\left[\sin(z_j) + \frac{kk_p + \cos(z_j)}{\sin(z_j)}(\cos(z_j) + 1)\right]$$
$$\Rightarrow w_j = \frac{z_j}{kL}\,\frac{(1 + kk_p)[1 + \cos(z_j)]}{\sin(z_j)}. \quad \text{(A.3)}$$

Now, following the same procedure used in the proof of Lemma A.2 we obtain

$$E_3(m, n) = \frac{(1 + kk_p)^2[\cos(z_m) - \cos(z_n)]}{kT[1 - \cos(z_m)]\,[1 - \cos(z_n)]}.$$

Now, since $k_p \neq -(1/k)$, then $(1 + kk_p)^2 > 0$. Also, since z_m, $z_n \neq 2(l - 1)\pi$, $l = 1, 2, 3, \ldots$, then $1 - \cos(z_m) > 0$ and $1 - \cos(z_n) > 0$. Thus, from the previous expression for $E_3(m, n)$ it is clear that

$$\text{sgn}[E_3(m, n)] = \text{sgn}[T] \cdot \text{sgn}[\cos(z_m) - \cos(z_n)].$$

This completes the proof of the lemma. ♣

Proof of Lemma 4.3

i) First we show that $b_j < -(T/k)$ for odd values of j. Recall from Fig. 6 that z_j is either in the first or second quadrant for odd values of j. Thus, $\sin(z_j) > 0$ for $j = 1, 3, 5, \ldots$. For $-(1/k) < k_p < 1/k$ and $\cos(z_j) < 1$, we have

$$\cos(z_j)(kk_p - 1) > kk_p - 1$$
$$\Rightarrow 1 + kk_p\cos(z_j) > \frac{T}{L}\,z_j\sin(z_j) \quad \text{[using (4.6)]}$$
$$\Rightarrow \frac{1 + kk_p\cos(z_j)}{\sin(z_j)} > \frac{T}{L}z_j \quad [\text{since } \sin(z_j) > 0]$$
$$\Rightarrow \sin(z_j) + \frac{T}{L}\,z_j\cos(z_j) > \frac{T}{L}\,z_j \quad \text{[using (A.1)]}$$
$$\Rightarrow b_j = -\frac{L}{kz_j}\left[\sin(z_j) + \frac{T}{L}\,z_j\cos(z_j)\right] < -\frac{T}{k}.$$

$$-\frac{kT}{L^2}\,E_1(m, n) = \frac{[kk_p + \cos(z_n)][1 + kk_p\cos(z_m)] - [kk_p + \cos(z_m)]\,[1 + kk_p\cos(z_n)]}{z_m z_n\sin(z_m)\sin(z_n)}$$
$$= \frac{[(kk_p)^2 - 1][\cos(z_m) - \cos(z_n)]}{z_m z_n\sin(z_m)\sin(z_n).}$$

Next, we show that $b_j < b_{j+2}$ for odd values of j. Since in this case, i.e., for $k_p \in (-(1/k), 1/k)$, $z_j \neq l\pi$ for odd values of j, from Lemma A.1 we have

$$E_1(j, j+2) := b_j - b_{j+2} = \frac{L^2}{kT} \frac{[1-(kk_p)^2][\cos(z_j) - \cos(z_{j+2})]}{z_j z_{j+2} \sin(z_j) \sin(z_{j+2})}.$$

Now, since $-(1/k) < k_p < 1/k$ then $1-(kk_p)^2 > 0$. We also know that $z_j > 0$ and $\sin(z_j) > 0$ for odd values of j. Then, from the previous expression for $E_1(j, j+2)$ and recalling that $T > 0$, we have

$$\text{sgn}[E_1(j, j+2)] = \text{sgn}[\cos(z_j) - \cos(z_{j+2})].$$

From Remark A.1 we know that $\cos(z_j) < \cos(z_{j+2})$. Then

$$\text{sgn}[E_1(j, j+2)] = -1$$
$$\Rightarrow E_1(j, j+2) = b_j - b_{j+2} < 0$$

and $b_j < b_{j+2}$ for odd values of j. Thus, we have shown that

$$b_j < b_{j+2} < -\frac{T}{k} \quad \text{for odd values of } j.$$

ii) We now show that $b_j > T/k$ for even values of j. From Fig. 6 we see that z_j is either in the third or fourth quadrant for even values of j. Thus, $\sin(z_j) < 0$ in this case. Since $\cos(z_j) > -1$ and $-1 < kk_p < 1$ we have

$$\cos(z_j)(1+kk_p) > -(1+kk_p)$$
$$\Rightarrow 1 + kk_p\cos(z_j) > -\frac{T}{L} z_j \sin(z_j) \quad \text{[using (4.6)]}$$
$$\Rightarrow \frac{1+kk_p\cos(z_j)}{\sin(z_j)} < -\frac{T}{L} z_j \quad [\text{since } \sin(z_j) < 0]$$
$$\Rightarrow \sin(z_j) + \frac{T}{L} z_j \cos(z_j) < -\frac{T}{L} z_j \quad \text{[using (A.1)]}$$
$$\Rightarrow b_j = -\frac{L}{kz_j}\left[\sin(z_j) + \frac{T}{L} z_j \cos(z_j)\right] > \frac{T}{k}.$$

Note that as $j \to \infty$, $z_j \to (j-1)\pi$. Then, $b_j \to T/k$.

iii) It only remains for us to show the properties of the parameter v_j when j takes on odd values. From (A.2) we have

$$v_j = \frac{z_j}{kL} \frac{(1-kk_p)[1-\cos(z_j)]}{\sin(z_j)}.$$

Now, since $-1 < kk_p < 1$ then $1 - kk_p > 0$. Also note that $1-\cos(z_j) > 0$. Moreover, when j takes on odd values then $\sin(z_j) > 0$. Thus we conclude that $v_j > 0$ for odd values of j. We now make use of Lemma A.2 to determine the sign of the quantity

$$E_2(j, j+2) := v_j - v_{j+2}.$$

Since $k_p \neq 1/k$ and $z_j \neq l\pi$ for odd values of j, the conditions in Lemma A.2 are satisfied and we obtain

$$\text{sgn}[E_2(j, j+2)] = \text{sgn}[T] \cdot \text{sgn}[\cos(z_j) - \cos(z_{j+2})].$$

We mentioned earlier that for odd values of j we have $\cos(z_j) < \cos(z_{j+2})$ and we also have $T > 0$ for an open-loop stable plant. Then, $\text{sgn}[E_2(j, j+2)] = -1$, so that $E_2(j, j+2) = v_j - v_{j+2} < 0$. Thus, we conclude that

$$0 < v_j < v_{j+2} \quad \text{for odd values of } j.$$

This completes the proof of the lemma. ♣

A. Proof of Lemma 4.4

i) We first consider the case of odd values of j. The proof follows from substituting (A.1) into (4.14) since $z_j \neq l\pi$, for odd values of j:

$$b_j = -\frac{L}{kz_j}\left[\frac{1+kk_p\cos(z_j)}{\sin(z_j)}\right] \quad \text{[using (A.1)]}$$
$$= -\frac{L}{kz_j}\left[\frac{1+\cos(z_j)}{\sin(z_j)}\right] \quad [\text{since } kk_p = 1]$$
$$= -\frac{T}{k} \quad \text{[using (4.6)]}.$$

ii) Now, for even values of j from Fig. 7 we see that $z_j = (j-1)\pi$. Then, $\sin(z_j) = 0$ and $\cos(z_j) = -1$ in this case. Thus, from (4.14) we conclude that $b_j = T/k$ for even values of j. This completes the proof of this lemma. ♣

B. Proof of Lemma 4.5

First, we make a general observation regarding the roots z_j, $j = 1, 2, 3, \ldots$ when the parameter k_p is inside the interval $(1/k, 1/k[(T/L)\alpha_1\sin(\alpha_1) - \cos(\alpha_1)])$. From Fig. 8(a), we see that these roots lie either in the first or second quadrant. Then

$$\sin(z_j) > 0 \quad \text{for } j = 1, 2, 3, \ldots. \tag{A.4}$$

i) We now consider the case of odd values of j. Since $k_p > 1/k$ and $\cos(z_j) < 1$ we have

$$\cos(z_j)(kk_p - 1) < kk_p - 1$$
$$\Rightarrow 1 + kk_p\cos(z_j) < \frac{T}{L} z_j \sin(z_j) \quad \text{[using (4.6)]}$$
$$\Rightarrow \sin(z_j) + \frac{T}{L} z_j\cos(z_j) < \frac{T}{L} z_j \quad \text{[using (A.1) and (A.4)]}$$
$$\Rightarrow b_j = -\frac{L}{kz_j}\left[\sin(z_j) + \frac{T}{L} z_j\cos(z_j)\right] > -\frac{T}{k}.$$

We now show that $b_j > b_{j+2}$. Since $z_j \neq l\pi$ for odd values of j, from Lemma A.1 we have

$$E_1(j, j+2) = b_j - b_{j+2} = \frac{L^2}{kT} \frac{[1-(kk_p)^2][\cos(z_j) - \cos(z_{j+2})]}{z_j z_{j+2}\sin(z_j)\sin(z_{j+2})}.$$

Now, since $k_p > 1/k$ then $1-(kk_p)^2 < 0$. We also know that $z_j > 0$ and $\sin(z_j) > 0$. Then, from the previous expression for $E_1(j, j+2)$ we have

$$\text{sgn}[E_1(j, j+2)] = -\text{sgn}[\cos(z_j) - \cos(z_{j+2})].$$

From Remark A.1, we have that $\cos(z_j) < \cos(z_{j+2})$. Then

$$\text{sgn}[E_1(j, j+2)] = 1$$
$$\Rightarrow E_1(j, j+2) = b_j - b_{j+2} > 0$$

and $b_j > b_{j+2}$ for odd values of j. Thus, we have shown that

$$b_j > b_{j+2} > -\frac{T}{k} \quad \text{for odd values of } j.$$

ii) We now consider the case of even values of the parameter j. Since $\cos(z_j) > -1$ and $k_p > 1/k$ we have

$$[\cos(z_j)+1](1+kk_p) > 0$$
$$\Rightarrow 1 + kk_p\cos(z_j) > -\frac{T}{L} z_j \sin(z_j) \quad \text{[using (4.6)]}$$
$$\Rightarrow \sin(z_j) + \frac{T}{L} z_j \cos(z_j) > -\frac{T}{L} z_j \quad \text{[using (A.1) and (A.4)]}$$
$$\Rightarrow b_j = -\frac{L}{kz_j}\left[\sin(z_j) + \frac{T}{L} z_j \cos(z_j)\right] < \frac{T}{k}.$$

We now show that $b_j < b_{j+2}$ for this case. We know that $z_j \neq l\pi$ for even values of j. Then, from Lemma A.1, we have

$$E_1(j, j+2) = b_j - b_{j+2} = \frac{L^2}{kT}\,\frac{[1-(kk_p)^2][\cos(z_j)-\cos(z_{j+2})]}{z_j z_{j+2}\sin(z_j)\sin(z_{j+2})}.$$

Once more since $k_p > 1/k$ then $1-(kk_p)^2 < 0$. We also know that $z_j > 0$ and $\sin(z_j) > 0$. Then, from the previous expression for $E_1(j, j+2)$ we have

$$\text{sgn}[E_1(j, j+2)] = -\text{sgn}[\cos(z_j)-\cos(z_{j+2})].$$

From Remark A.2, we have that $\cos(z_j) > \cos(z_{j+2})$ for even values of the parameter j. Using this fact we obtain

$$\text{sgn}[E_1(j, j+2)] = -1$$
$$\Rightarrow E_1(j, j+2) = b_j - b_{j+2} < 0$$

and $b_j < b_{j+2}$ for even values of j. Thus, we have shown that

$$b_j < b_{j+2} < \frac{T}{k} \quad \text{for even values of } j.$$

iii) We now consider the properties of the parameter w_j. From (A.3), we have

$$w_j = \frac{z_j}{kL}\,\frac{(1+kk_p)[1+\cos(z_j)]}{\sin(z_j)}.$$

Clearly since $k_p > 1/k$ then $1 + kk_p > 0$. Also notice that $1+\cos(z_j) > 0$. Thus, since $\sin(z_j) > 0$ we conclude that $w_j > 0$ for even values of the parameter j. We now invoke Lemma A.3 and evaluate the function $E_3(m, n)$ at $m = j$, $n = j+2$

$$E_3(j, j+2) = w_j - w_{j+2}.$$

Since $k_p \neq -(1/k)$ and $z_j \neq l\pi$ for even values of j, then we have

$$\text{sgn}[E_3(j, j+2)] = \text{sgn}[T]\cdot\text{sgn}[\cos(z_j)-\cos(z_{j+2})].$$

We know from Remark A.2 that $\cos(z_j) > \cos(z_{j+2})$ for even values of j, and also that $T > 0$. Then, $\text{sgn}[E_3(j, j+2)] = 1$, so that $E_3(j, j+2) = w_j - w_{j+2} > 0$. Thus, we have shown that

$$w_j > w_{j+2} > 0 \quad \text{for even values of } j.$$

iv) We show first that $b_1 < b_2$. Since $z_1, z_2 \neq l\pi$, from Lemma A.1 we have

$$E_1(1, 2) = b_1 - b_2 = \frac{L^2}{kT}\,\frac{[1-(kk_p)^2][\cos(z_1)-\cos(z_2)]}{z_1 z_2 \sin(z_1)\sin(z_2)}.$$

We know that $\sin(z_1) > 0$ and $\sin(z_2) > 0$. Moreover, since $k_p > 1/k$ we obtain the following:

$$\text{sgn}[E_1(1, 2)] = -\text{sgn}[\cos(z_1)-\cos(z_2)].$$

As we can see from Fig. 8(a), both z_1 and z_2 are in the interval $(0, \pi)$ and $z_1 < z_2$. Since the cosine function is monotonically decreasing in $(0, \pi)$ then $\cos(z_1) > \cos(z_2)$. Thus

$$\text{sgn}[E_1(1, 2)] = -1$$
$$\Rightarrow E_1(1, 2) = b_1 - b_2 < 0.$$

Hence, we have $b_1 < b_2$. Finally we show that $w_1 > w_2$. To do so, we invoke Lemma A.3 and evaluate the function $E_3(m, n)$ at $m = 1$, $n = 2$:

$$E_3(1, 2) = w_1 - w_2.$$

Since $k_p \neq -(1/k)$ and $z_1, z_2 \notin \{0, \pi\}$, the conditions in Lemma A.3 are satisfied and we obtain

$$\text{sgn}[E_3(1, 2)] = \text{sgn}[T]\cdot\text{sgn}[\cos(z_1)-\cos(z_2)].$$

We already pointed out that $\cos(z_1) > \cos(z_2)$ and since $T > 0$ we have

$$\text{sgn}[E_3(1, 2)] = 1$$
$$\Rightarrow E_3(1, 2) = w_1 - w_2 > 0.$$

Thus, $w_1 > w_2$ and this completes the proof of the lemma. ♣

APPENDIX B
PROOF OF LEMMA 4.7

Let us define the function $f: (0, \pi) \times \mathcal{R} \to \mathcal{R}$ by

$$f(z, k_p) \triangleq \frac{kk_p + \cos(z)}{\sin(z)}.$$

Consider $k_{p1}, k_{p2} \in \mathcal{R}$ such that $k_{p1} < k_{p2}$. Then, for any $z \in (0, \pi)$, we have

$$kk_{p1} + \cos(z) < kk_{p2} + \cos(z)$$
$$\Rightarrow \frac{kk_{p1}+\cos(z)}{\sin(z)} < \frac{kk_{p2}+\cos(z)}{\sin(z)} \quad [\text{since } \sin(z) > 0]$$
$$\Rightarrow f(z, k_{p1}) < f(z, k_{p2}).$$

Thus for any fixed $z \in (0, \pi)$, $f(z, k_p)$ is monotonically increasing with respect to k_p. Hence, for $k_p < -(1/k)$ we have

$$f(z, k_p) < f\left(z, -\frac{1}{k}\right) \qquad \forall\, z \in (0, \pi).$$

This means that if the line $(T/L)z$ does not intersect the curve $f(z, -(1/k))$ in $z \in (0, \pi)$, then it will not intersect any other curve $f(z, k_p)$ in $z \in (0, \pi)$.

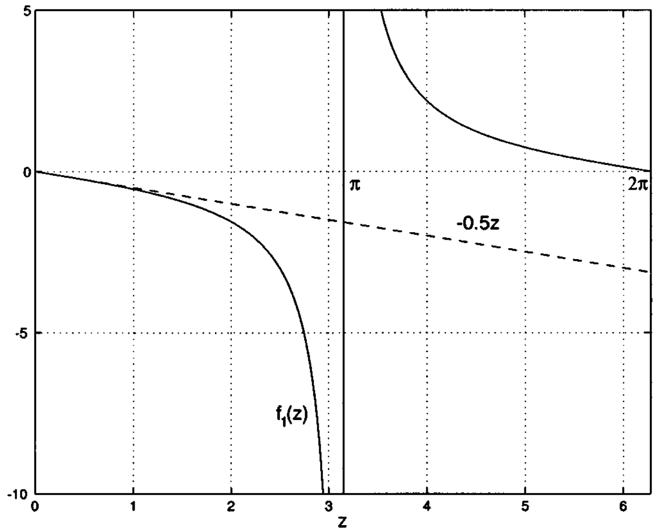

Fig. 10. Plot of the curve $f_1(z, -(1/k))$ and the line $(T/L)z$.

Observe that $\forall z \in (0, \pi)$

$$f\left(z, -\frac{1}{k}\right) = \frac{-1+\cos(z)}{\sin(z)} = -\tan\left(\frac{z}{2}\right).$$

Accordingly, define a continuous extension of $f(z, -(1/k))$ to $[0, \pi)$ by

$$f_1\left(z, -\frac{1}{k}\right) = -\tan\left(\frac{z}{2}\right).$$

Clearly, the curve $f_1(z, -(1/k))$ intersects the line $(T/L)z$ at $z = 0$. This is depicted in Fig. 10. Also, note that the slope of the tangent to $f_1(z, -(1/k))$ at $z = 0$ is given by

$$\left.\frac{df_1}{dz}\right|_{z=0} = -\frac{1}{2}\sec^2\left(\frac{z}{2}\right)\Big|_{z=0} = -\frac{1}{2}.$$

Clearly, if this slope is less than or equal to T/L then we are guaranteed that no further intersections will take place in the interval $(0, \pi)$. Since $f_1(z, -(1/k)) = f(z, -(1/k))$ on $(0, \pi)$, it follows that if $-0.5 \leq T/L$, then the curve $f(z, -(1/k))$ will not intersect the line $(T/L)z$ in the interval $(0, \pi)$. This completes the proof. ♣

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